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STABILIZATION OF THE STATIONARY MOTIONS OF NON-HOLONOMIC MECHANICAL SYSTEMS*

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The possible stabilization of the unstable stationary motions of a non-holonomic system is studied from the standpoint of general control theory /1, 2/. As distinct from the case previously considered /3/, when forces of a certain structure are applied with respect to both positional and cyclical coordinates, the stabilization is obtained here by applying control forces only with respect to the cyclical coordinates /4/; the control forces may be applied with respect to some or all of the cyclical coordinates, and depend on the positional coordinates, the velocities, and the corresponding cyclical momenta. It is shown that, just as in the case of holonomic systems /5, 6/, depending on the control properties of the corresponding linear subsystem, the stationary motions, whether stable or unstable, can be stabilized, up to asymptotic stability with respect to all the phase variables, or asymptotic stability with respect to some of the phase variables and stability with respect to the remaining variables. The type of stabilization with respect to the given phase variables depends on the Lyapunov transformations which are needed in order to reduce the critical cases obtained to singular cases /7, 8/.

1. Let the non-holonomic constraints on a system under the action of potential forces be

$$q_\mu^{\cdot} = B_{\mu\rho}(q)\, q_\rho^{\cdot} \tag{1.1}$$

Here and henceforth,

$$\mu = 1, 2, \ldots, m;\ \varkappa = 1, 2, \ldots, m - l$$
$$\eta = m - l + 1, \ldots, m;\quad \alpha, \beta, \nu = m + 1, m + 2, \ldots, k$$
$$i, j = m + k + 1, m + k + 2, \ldots, n;\ \omega, \rho, s = m + 1, m + 2, \ldots, n$$

Summation is performed with respect to twice repeated indices.

As in /3/, we shall describe the state of the system by the Routh variables $q_\mu, q_\mu^{\cdot}, q_i, q_i^{\cdot}, q_\alpha, p_\alpha$. We first consider the case when there is a manifold of stationary motions whose dimensionality is not less than the sum of the number of cyclical coordinates and the number of non-holonomic constraints of a general type /9/: let the latter number be independent of the cyclical velocities, i.e., $B_{\eta\alpha} \equiv 0$, and we have the conditions

$$\begin{gathered}
\frac{\partial}{\partial q_\varkappa}(T_0 - \Pi) = 0,\quad \frac{\partial}{\partial q_\varkappa} B_{\mu\rho} = 0,\quad \theta_{\varkappa\beta}\Omega_{\varkappa\alpha\nu} = 0 \\
\frac{\partial R}{\partial q_\alpha} = 0,\quad \frac{\partial}{\partial q_\alpha} B_{\nu\rho} = 0,\quad \frac{\partial}{\partial q_\alpha}(\theta_{\mu\rho}\Omega_{\mu s\omega}) = 0 \\
R = {}^1/_2\, a_{ij}{}^* q_i^{\cdot} q_j^{\cdot} + \gamma_{\alpha i} p_\alpha q_i^{\cdot} - {}^1/_2\, b_{\alpha\beta} p_\alpha p_\beta - \Pi(q)
\end{gathered} \tag{1.2}$$

Here, T_0 is the kinetic energy of the system when the constraints (1.1) are ignored, $\Pi(q)$ is the potential energy, R is Routh's function and $\theta_{\mu\rho}\Omega_{\mu s\omega}$ are the coefficients of the terms of non-holonomicity in Voronets's equations.

The equations of the perturbed motion in the neighbourhood of stationary motion

$$q_\eta = q_{\eta 0},\ q_i = q_{i0},\ q_i^{\cdot} = 0,\ p_\alpha = c_\alpha = \text{const} \tag{1.3}$$

have the form /3/

$$\begin{gathered}
s^{\cdot} = Bx^{\cdot} + \Phi_1(x, s, x^{\cdot}),\ y^{\cdot} = Nx^{\cdot} + \Phi_2(x, s, y, x^{\cdot}) \\
Ax^{\cdot\cdot} + \Gamma y^{\cdot} + (D_1 + G_1)\, x^{\cdot} + Mx + Hy + Es = \Phi_3(x, s, y, x^{\cdot})
\end{gathered}$$

where y, x, s are respectively the disturbance vectors of the cyclical momenta p_α, the positional coordinates q_i and the coordinates q_η, whose velocities are independent by virtue of (1.1). The coefficients of these equations are constant matrices, expressible in a well-known way /3/ in terms of the coefficients of the constraints, the Routh function, and the terms of non-holonomicity for the stationary motion (1.3); Φ_1, Φ_2, Φ_3 are non-linear vector functions

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whose expansions start with a second order term, while

$$\Phi_1(x, s, 0) = \Phi_2(x, s, y, 0) \equiv 0$$

Making the change of variables /10/ $z = s - Bx$ in these equations, we reduce them to the normal form

$$\begin{gathered}z^{\cdot} = \Phi_1(x, z + Bx, x_1),\ y^{\cdot} = Nx_1 + \Phi_2(x, z + Bx, y, x_1) \\ x^{\cdot} = x_1,\ x_1^{\cdot} = -A^{-1}[\Gamma y^{\cdot} + (D_1 + G_1)x_1 + (M + EB)x + \\ Hy + Ez - \Phi_3(x, z + Bx, y, x_1)\end{gathered} \tag{1.4}$$

2. We will consider the stabilization of stationary motions by applying control signals only with respect to the cyclical coordinates. We first consider the case when the control forces are applied with respect to all the cyclical coordinates and depend on the positional coordinates, the velocities, and all the cyclical momenta. We consider the linear subsystem

$$y^{\cdot} = Nx_1,\ x^{\cdot} = x_1,\ x_1^{\cdot} = -A^{-1}[\Gamma y^{\cdot} + (D_1 + G_1)x_1 + (M + EB)x + Hy] \tag{2.1}$$

We apply to this subsystem the vector of control forces $v\ (\dim v = k)$ with respect to the cyclical coordinates and stabilize the stationary motion $y = x = x_1 = 0$ up to asymptotic stability with respect to the variables y, x, x_1. We take as the performance criterion of the transient the functional

$$\int_{t_0}^{\infty} \omega(x, x_1, y, v)\, dt \tag{2.2}$$

where ω is the sum of the positive definite quadratic forms with respect to the variables y_α, x_i, x_{1i} and the control v_α.

Put

$$Q' = \{I, 0, -A^{-1}\Gamma\} \tag{2.3}$$

$$P = \begin{Vmatrix} 0 & 0 & N \\ 0 & 0 & I \\ -A^{-1}H & -A^{-1}(M + EB) & -A^{-1}(\Gamma N + D_1 + G_1) \end{Vmatrix}$$

where I is the identity matrix and 0 the zero matrix.

By control theory for linear systems /1, 2/, if the rank of the matrix

$$W = \{Q, PQ, \ldots, P^{2(n-m)-k-1}Q\} \tag{2.4}$$

is equal to the order of system (2.1), the problem of optimal stabilization will have a solution. The coefficients of the linear control are then

$$v = L_1 x + L_2 x_1 + L_3 y \tag{2.5}$$

and can be uniquely determined from the chosen optimal Lyapunov quadratic function V° /2/. The usual algebraic equations are obtained for the coefficients of V°. In certain cases /2, 5/, the coefficients of V° and of the control (2.5) can be found analytically.

If the linear control (2.5) can thus be found, then all the roots of the characteristic equation

$$\Delta_1(\lambda) = \det\begin{Vmatrix} -L_1 & -L_2 - N & I\lambda - L_3 \\ I\lambda & I & 0 \\ -A^{-1}(M + EB + \Gamma L_1) & I\lambda - A^{-1}(\Gamma N + \Gamma L_2 + D_1 + G_1) & -A^{-1}(H + \Gamma L_3) \end{Vmatrix} = 0$$

for the controlled subsystem (2.1) will have negative real parts.

Let us examine the stability of the zero solution of the complete system of Eqs.(1.4) under the action of the controls (2.5) which solve the optimal stabilization problem for the zero solution of the controlled subsystem (2.1):

$$\begin{gathered}z^{\cdot} = \Phi_1(x, z + Bx, x_1),\ y^{\cdot} = (N + L_2)x_1 + L_1 x + L_3 y + \\ \Phi_2(x, z + Bx, y, x_1),\ x^{\cdot} = x_1 \\ x_1^{\cdot} = -A^{-1}[(\Gamma N + \Gamma L_2 + D_1 + G_1)x_1 + (M + EB + \Gamma L_1)x + \\ (H + \Gamma L_3)y + Ez - \Phi_3 + \Gamma\Phi_2]\end{gathered} \tag{2.6}$$

The characteristic equation of the first approximation of this system

$$\lambda^l \Delta_1(\lambda) = 0$$

has l zero roots, where l is the number of constraints of general type, while the remaining roots lie in the left-hand half-plane.

We will now show that this critical case can be reduced to a singular case /7, 8/.

Since $\Delta_1(0) \neq 0$, there exist implicit functions $u_1(z)$, $u_2(z)$, given by the system of equations

$$L_1 u_1(z) + L_3 u_2(z) = 0 \tag{2.7}$$
$$(M + EB + \Gamma L_1) u_1(z) + (H + \Gamma L_3) u_2(z) + Ez - \Phi_3^* = 0$$
$$(\Phi_3^* = \Phi_3(u_1(z), z + Bu_1(z), u_2(z), 0))$$

We make the Lyapunov change of variables /7/

$$x = \zeta + u_1(z), \quad y = \eta + u_2(z) \tag{2.8}$$

System (2.6) then becomes

$$z^{\cdot} = \Phi_1, \quad \eta^{\cdot} = (N + L_2) x_1 + L_1\zeta + L_3\eta + \Phi_2^0 \tag{2.9}$$
$$\zeta^{\cdot} = x_1 - (\partial u_1/\partial z)\Phi_1$$
$$x_1^{\cdot} = -A^{-1}[(\Gamma N + \Gamma L_2 + D_1 + G_1) x_1 + (M + EB + \Gamma L_1)\zeta + (H + \Gamma L_3)\eta - \Phi_3^{\circ}]$$
$$(\Phi_3^{\circ} = \Phi_3 - \Phi_3^* - \Gamma\Phi_2, \quad \Phi_2^{\circ} = \Phi_2 - (\partial u_2/\partial z)\Phi_1)$$

where Φ_2, Φ_3 vanish for $\zeta = \eta = x_1 = 0$.

In short, the system of equations of the disturbed motion has been reduced to the form (2.9), for which the conditions of the Lyapunov-Malkin theorem /7, 8/ on the singular case of several zero roots, are satisfied. By this theorem, the stationary motion (1.3) is asymptotically stable with respect to the variables ζ, η, x_1, and stable with respect to z. We thus have the following theorem.

Theorem 1. If the rank of the matrix W of (2.4) is $2(n - m) - k$, then the stationary motion (1.3) can be stabilized by applying the control forces (2.5) with respect to the cyclical coordinates only.

In this case, taking into account the substitutions (2.8) we obtain asymptotic stability with respect to the position velocities $q_i^{\cdot}$ and stability with respect to the coordinates q_i, q_η and cyclic momenta p_α.

Notes. 1°. Theorem 1 also holds if, for stabilization of the zero solution of the complete system, we take the control vector

$$v = L_1 x + L_2 x_1 + L_3 y + \Phi_4(x, x_1, y)$$

where Φ_4 is a non-linear vector function whose expansion starts with second-order terms in x, x_1, y. In this case, we must add to Eqs. (2.7) the respective terms Φ_4^* and $\Gamma\Phi_4^*$, similar to Φ_3^*.

2°. If the last equation of system (2.6) contains no unattached critical variables z, we obtain asymptotic stability with respect to the variables $q_i, q_i^{\cdot}, p_\alpha$, and stability with respect to the coordinates q_η. For, in this case there is no need to make the change of variables (2.8), and system (2.6) at once satisfies the conditions of the Lyapunov-Malkin theorem about the singular case.

3°. Given our structure of the control (2.5), as distinct from /3/, we can if necessary compensate the dissipative forces which act on the cyclical coordinates and depend on the positional and cyclical velocities. In this case it suffices to take account of the respective terms $L_2^{\circ} x_1 + L_3^{\circ} y + Q_0$ in the control, where Q_0 is a constant matrix.

3. We now consider the stabilization of stationary motions by applying controls with respect to only some of the cyclical variables.

Let y_1 ($\dim y_1 = k_1 < k$) be the part of vector y with respect to which no control is applied, while y_2 ($\dim y_2 = k - k_1$) is the part with respect to which control v_2 is applied. We then have to make the change of variables $w_1 = y_1 - Nx_1$, and the critical variables will be z and w_1. The linear controlled subsystem is

$$y_2^{\cdot} = N_2 x_1 + v_2, \quad x^{\cdot} = x_1 \tag{3.1}$$
$$x_1^{\cdot} = -A^{-1}[(\Gamma N + D_1 + G_1) x_1 + \Gamma v_2 + (M + EB + H_1 N_1) x + H_2 y_2]$$

If the rank of the matrix

$$W = \{Q, PQ, \ldots, P^{2(n-m)-k-k_1-1}Q\} \tag{3.2}$$

(where Q and P have the form (2.3) when H is replaced by H_2, N by N_2, and M by $M + H_1N_1$) is equal to $2(n - m) - k - k_1$, then the control (2.5) solves the problem of optimal stabilization for subsystem (3.1).

The characteristic equation of the first approximation of the complete system has $l + k_1$ zero roots. Then, in the same way as above, we can use the Lyapunov replacement

$$x = \zeta + u_1(z, w_1), \quad y_2 = \eta + u_2(z, w_1)$$

to reduce the system of complete equations of the perturbed motion to the singular case of $l + k_1$ zero roots.

We have thus proved:

Theorem 2. If the rank of matrix W of (3.2) is equal to the order of the linear controlled subsystem (3.1), then the stationary motion (1.3) can be stabilized by applying the control forces (2.5) with respect to the part of the cyclical coordinates corresponding to the momenta y_2.

4. Now let the equations of the constraints depend on the cyclical velocities, i.e., $B_{\eta\alpha} \neq 0$, while in the Routh variables the equations of the constraints of general type have the form /1, 2/

$$q_\eta^{\cdot} = (B_{\eta i} - B_{\eta\alpha}\gamma_{\alpha i})\, q_i^{\cdot} + B_{\eta\alpha} b_{\alpha\beta} p_\beta$$

Let conditions (1.2) and the following conditions for the existence of stationary motions of type (1.3) be satisfied:

$$(B_{\eta\alpha} b_{\alpha\beta})^\circ c_\beta = 0$$

In this case, the equations of the first approximation of the equations of the perturbed motion, both for the constraints of general type, and for the cyclical momenta, will in general contain all the variables $x, x^{\cdot}, s, y$.

For simplicity, we consider the case /12/ when

$$B_{\eta\alpha} \partial R_0 (q, p) / \partial q_\eta \equiv 0$$

where

$$R_0 (q, p) = -\Pi (q) - {}^1/_2 b_{\alpha\beta} p_\alpha p_\beta$$

is the part of the Routh function which does not contain positional velocities. In this case the equations of the perturbed motion are /12/

$$\begin{gathered} s^{\cdot} = Bx_1 + Kx + \Phi s + Ty + \Phi_1^\circ \\ y^{\cdot} = Nx_1 + \Phi_2^\circ,\ x^{\cdot} = x_1,\ x_1^{\cdot} = -A^{-1} [(D + G)\, x_1 + \Gamma y^{\cdot} + \\ Mx + Es + Hy - \Phi_3^\circ] \end{gathered} \tag{4.1}$$

where the coefficients are expressible in the usual way /11, 12/ in terms of the coefficients of the constraints, the Routh function, and the terms of non-holonomicity.

Assume that $T = (B_{\eta\alpha} b_{\alpha\beta}) \neq 0$. We apply with respect to the cyclical coordinates the control force vector

$$v = L_1 x + L_2 x_1 + L_3 y + L_4 s + \Phi_4 (x, x_1, y, s) \tag{4.2}$$

and consider the matrix

$$W = \{Q_1, P_1 Q_1, \ldots, P_1^{2(n-m)-k+l-1} Q_1\} \tag{4.3}$$

$$Q' = \{0, I, 0, -A^{-1}\Gamma\}$$

$$P_1 = \begin{Vmatrix} \Phi & T & K & B \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -A^{-1}E & -A^{-1}H & -A^{-1}M & -A^{-1}(D + G + \Gamma N) \end{Vmatrix}$$

The theorem on stabilization with respect to the first approximation /2/ then leads to:

Theorem 3. If the rank of the matrix W of (4.2) is equal to the order of system (4.1), i.e., is $2(n - m) - k + l$, then the stationary motion (1.3) can be stabilized to a first approximation by applying the control forces (4.2) with respect to the cyclical coordinates only. The undisturbed motion is then asymptotically stable with respect to the positional coordinates q_η, q_i, the velocities $q_i^{\cdot}$, and the cyclical momenta p_α, with any non-linear terms.

Note. The controls obtained by Theorems 1 and 2 will not in general be optimal for the system of Eqs.(1.4) in the case of our functional which has a quadratic integrand. These controls will only solve the stabilization problem for system (1.4). However, by starting the optimal stabilization problem for the linear controlled subsystem, we can determine whether the stabilization problem is solvable for the complete system. We then find at the same time the structure of the control forces.

The controls obtained in accordance with Theorem 3 will also be optimal for the complete system of equations if we take as the functional

$$\int_{t_0}^{\infty} (\omega + \omega_1)\, dt$$

where ω is the sum of the positive definite quadratic forms with respect to the variables x, x_1, y, s, and the control v

$$\omega_1 = \frac{\partial V^\circ}{\partial y} \Phi_2^\circ + \frac{\partial V^\circ}{\partial x_1} (\Phi_3^\circ - \Gamma \Phi_2^\circ) + \frac{\partial V^\circ}{\partial s} \Phi_1^\circ$$

where V° is the optimal Lyapunov function, which is found by solving the problem of optimal stabilization of the first approximation of system (4.1) for the quadratic functional (2.2).

Example. The unstable rotation of a disc on a rough horizontal plane /13/ has been stabilized by applying with respect to the cyclical coordinate φ a moment which depends on the positional velocity $\theta^\cdot$ /3/. It is then necessary to apply a dissipative force with respect to the positional coordinate.

We now consider the possible stabilization of the unstable stationary motions of the disc, regardless of whether dissipative forces acting on the positional coordinate are present.

The Lagrange function L, formed in the context of non-holonomic constraints $x^\cdot = a\varphi^\cdot \cos\psi$, $y^\cdot = a\varphi^\cdot \sin\psi$, is given by

$$L = {}^1/_2\,[A^\circ\theta^{\cdot 2} + (C + ma^2)(\varphi^\cdot - \psi^\cdot \sin\theta)^2 + A\psi^{\cdot 2}\cos^2\theta] - mga\cos\theta,$$
$$A^\circ = A + ma^2$$

Here, m is the mass of the disc, a is its radius, and A and C are the equatorial and polar moments of inertia /13/.

We write the equations of the perturbed motion /3/ in the linear approximation in the neighbourhood of a stationary motion $\theta = \theta_0$, $p_1 = c_1 = \text{const}$, $p_2 = c_2 = \text{const}$.

Putting $\theta = \theta_0 + \eta$, $p_1 = c_1 + y_1$, $p_2 = c_2 + y_2$, we obtain

$$y_1^\cdot = N_1\eta^\cdot,\quad y_2^\cdot = N_2\eta^\cdot$$
$$A^\circ\eta^{\cdot\cdot} + h\eta^\cdot + M\eta + H_1y_1 + H_2y_2 = 0$$

where the constant coefficients depend on the choice of stationary motion /3/.

We apply to the disc, with respect to the angle of proper rotation and the precession, the controls v_1 and v_2, see (2.5), which depend linearly on the variables $z_1, z_2, \eta, \eta^\cdot$ ($z_\delta = y_\delta - N_\delta$, $\delta = 1, 2$). The matrices Q and P will then be

$$Q = \left\| \begin{matrix} I \\ 0 \end{matrix} \right\|,\quad P = \left\| \begin{matrix} 0 & 0 \\ P_1 & P_2 \end{matrix} \right\|;\quad P_1 = \left\| \begin{matrix} 0 & 0 \\ H_1^\circ & H_2^\circ \end{matrix} \right\|,\quad P_2 = \left\| \begin{matrix} 0 & 1 \\ M^\circ & h^\circ \end{matrix} \right\|$$
$$H_\delta^\circ = -H_\delta/A^\circ,\quad M^\circ = -(M + H_1N_1 + H_2N_2)/A^\circ,\quad h^\circ = -h/A^\circ$$

We first consider circular motion, for which c_1, c_2, θ_0 retain constant values. The rank of the matrix W is then equal to four, regardless of whether dissipation is present $(h \neq 0)$ or not $(h = 0)$. Consequently, the unstable circular motion can be stabilized by applying simultaneously with respect to the cyclical coordinates φ and ψ two linear control moments v_1 and v_2, which depend on the variable $z_1, z_2, \eta, \eta^\cdot$. The coefficients of these controls must satisfy the conditions that are implied by the Hurwitz criterion, whereby the roots of the characteristic equation are negative. It can be shown that spinning $(p_1 = c_1 = 0,\ p_2 = c_2 = \text{const},\ \theta_0 = 0)$ and wobbling on a straight line $(p_1 = c_1 = \text{const},\ p_2 = c_2 = 0,\ \theta_0 = 0)$ can similarly be stabilized by two controls v_1 and v_2 if $M \neq 0$. With this stabilization, the circular motion, spinning, and wobbling on a straight line, will be asymptotically stable with respect to $\theta, \theta', p_1, p_2$.

We shall now stabilize the circular motion by a control moment $v_2 = l_1\eta + l_2\eta^\cdot + l_3z_1$, applied with respect to the cyclical coordinate ψ only. In this case, z_1 will be a critical variable. Taking the subsystem consisting of the three remaining equations, we see that the rank $W = 3$ whether dissipation is present $(h \neq 0)$ or not $(h = 0)$.

On considering the complete system of equations, we see by Theorem 2 that the circular motion is asymptotically stable with respect to $\theta^\cdot$ and stable with respect to θ, p_1, p_2.

Similarly, the circular motion can be stabilized by the control moment $v_1 = l_1\eta + l_2\eta^\cdot - l_3z_1$, applied with respect to the cyclical coordinate φ.

On applying the control moment v_1 with respect to φ, the spinning can be stabilized regardless of whether or not dissipation is present. We then have asymptotic stability with respect to $p_1, \theta, \theta^\cdot$, and stability with respect to p_2, since in this case $H_2 = 0$ and there are no unattached critical variables y_2. For comparison with /3/, we write the conditions defining the coefficients of the control moment:

$$l_3 < 0,\quad l_2 > -l_1 > (A\Omega^2 + ma^2\Omega - mga)/\Omega$$

If, in the case of spinning, we apply the control moment with respect to the cyclical coordinate ψ, the system is uncontrolled to a first approximation.

Finally, take the stabilization of wobble along a straight line by means of one control. By adding a control v_2 with respect to the cyclical coordinate ψ, we find that the subsystem of the last three equations is controlled; for the complete system of equations we have asymptotic stability with respect to $p_2, \theta, \theta^\cdot$, and stability with respect to p_1, since the complete system of equations contains no unattached critical variables y_1. If the control is applied with respect to the cyclical coordinate φ, the system is uncontrolled to a first approximation.

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ON THE RELATIVE EQUILIBRIA OF A SATELLITE-GYROSTAT, THEIR BRANCHINGS AND STABILITY*

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The set of relative equilibria of a satellite-gyrostat in a Newtonian gravitational field is studied. The simple geometrical form of this set is described. The branching and stability of the equilibria of a symmetric gyrostat are considered. The results are represented by bifurcation diagrams, on which the degree of stability of the equilibria is distributed in accordance with a law whereby the stability changes at a fixed value of the gyrostatic moment.

1. In some problems of gyrostat dynamics in a Newtonian gravitational field /1-6/, the determination of the positions of relative equilibrium of the gyrostat amounts to finding the stationary values of the function

$$W = \frac{1}{2}\sum_{j=1}^{3}(3hA_j\gamma_j^2 - A_j\beta_j^2 - 2k_j\beta_j)$$

under the conditions

$$\pi_\gamma = \gamma_1^2 + \gamma_2^2 + \gamma_3^2 - 1 = 0,\ \pi_\beta = \beta_1^2 + \beta_2^2 + \beta_3^2 - 1 = 0 \qquad (1.1)$$
$$\pi_{\gamma\beta} = \gamma_1\beta_1 + \gamma_2\beta_2 + \gamma_3\beta_3 - 0$$

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